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On Centered and Weakly Centered Operators

VERN PAULSEN

Department of Mathematics, University of Houston, Houston, Texas 77204

AND

CARL PEARCY AND SRDJAN PETROVIĆ

Department of Mathematics, Texas A&M University, College Station, Texas 77843

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We show that if C is a contraction of spectral radius 1 then T is polynomially bounded if and only if the same is true of $T \otimes C$. As a corollary, we obtain that every invertible, centered, polynomially bounded operator is similar to a contraction. We also prove a structure theorem for weakly centered operators that satisfy a hypothesis of a technical nature. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $\mathscr{H}$ denote a separable, infinite-dimensional, complex Hilbert space and $\mathscr{L}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. An operator T in $\mathscr{L}(\mathscr{H})$ is said to be *polynomially bounded* (notation: $T \in (\mathrm{PB})(\mathscr{H})$ or $T \in (\mathrm{PB})$) if there exists an $M \geqslant 1$ such that

$$\|p(T)\| \leqslant M \sup\{|p(\zeta)| : |\zeta| = 1\} \qquad \forall \text{ polynomial } p, \tag{1}$$

and to be *power bounded* (notation: $T \in (\mathrm{PW})$) if (1) holds for every polynomial of the special form $p(\zeta) = \zeta^n$, where n is a positive integer. If $T \in (\mathrm{PB})$ (resp., $T \in (\mathrm{PW})$), then there is a smallest number $M \geqslant 1$ which satisfies (1) (resp., (1) restricted). This number will be called the *polynomial bound* (resp., the *power bound*) of T and will be denoted by $M_{\mathrm{pb}}(T)$ (resp., $M_{\mathrm{pw}}(T)$). (If $T \notin (\mathrm{PB})$ (resp., $T \notin (\mathrm{PW})$), we set M_{pb} (resp., M_{pw}) equal to $+\infty$.) Also an operator T in $\mathscr{L}(\mathscr{H})$ is said to be *similar to a contraction* (notation: $T \in (\mathrm{SC})$) if there exists an invertible operator S in $\mathscr{L}(\mathscr{H})$ such

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that $\|S^{-1}TS\| \leqslant 1$, and to be *completely polynomially bounded* (notation: $T \in (\mathrm{CPB})$) if there exists an $M \geqslant 1$ such that one has

$$\|(p_{ij}(T))\| \leqslant M \sup_{|\zeta|=1} \|(p_{ij}(\zeta))\| \\ \forall n \in \mathbb{N}, \forall \text{ family } \{p_{ij}\}_{i,j=1}^{n} \text{ of polynomials,} \tag{2}$$

where the operator $(p_{ij}(T))$ on the left side of (2) is an $n \times n$ matrix with operator entries acting, in the usual fashion, on the direct sum of n copies of $\mathscr{H}$, and $(p_{ij}(\zeta))$ denotes the obvious $n \times n$ complex matrix. If $T \in (\mathrm{CPB})$ then there is, once again, a smallest number $M \geqslant 1$ satisfying (2), called the *complete polynomial bound* of T and denoted by $M_{\mathrm{cpb}}(T)$. It is elementary that

$$(\mathrm{SC}) \subset (\mathrm{CPB}) \subset (\mathrm{PB}) \subset (\mathrm{PW}),$$

and it was proved by the first author [13] that $(\mathrm{SC})(\mathscr{H}) = (\mathrm{CPB})(\mathscr{H})$ and that

$$M_{\mathrm{cpb}}(T) = \min\{\|S\|\,\|S^{-1}\| : \|S^{-1}TS\| \leqslant 1\}, \qquad T \in (\mathrm{SC}). \tag{3}$$

We will use this result throughout the paper without further comment.

On the other hand, it was shown by Foguel [6] (see also [7]) that $(\mathrm{PB}) \neq (\mathrm{PW})$, and it is a difficult and interesting open question, posed explicitly by Halmos in [8], whether

$$(\mathrm{PB}) \subset (\mathrm{SC}). \tag{4}$$

For more information concerning this circle of ideas, one might consult [14].

The purpose of this paper is to make a contribution toward determining the correctness of (4) along the following lines. In [17], the third author showed that if $T \in (\mathrm{PB})(\mathscr{H})$, then T is the compression to a semi-invariant subspace of an operator $\tilde{T}$ acting on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ such that $\tilde{T} \in (\mathrm{PB})(\mathscr{K})$, the spectrum of $\tilde{T}$ is the unit circle in the complex plane, and $\tilde{T}$ is *weakly centered* (notation: $T \in (\mathrm{WC})$), meaning that $\tilde{T}\tilde{T}^*$ commutes with $\tilde{T}^*\tilde{T}$. (Said otherwise, $\tilde{T}$ is a dilation of T with certain nice properties.) It is an easy consequence of this theorem (cf. [17] for details) that (4) is valid if and only if

$$(\mathrm{PB}) \cap (\mathrm{WC})_{\mathrm{in}} \subset (\mathrm{SC}), \tag{5}$$

where $(\mathrm{WC})_{\mathrm{in}}$ denotes the class of invertible weakly centered operators. This naturally leads to the question: what is the structure of an invertible weakly centered operator? (It should be said at once that weakly centered operators were studied briefly by Campbell in [2, 3] under the name binormal operators.) In what follows we study the structure theory of the class (WC). Moreover, it turns out that there is another, related, class of operators, called the *centered* operators, about which much is known. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *centered* (notation: $T \in (\mathrm{C})$) if the doubly infinite sequence

$$\{..., T^n T^{*n}, ..., T^2 T^{*2}, TT^*, T^*T, T^{*2}T^2, ..., T^{*n}T^n, ...\}$$

consists of mutually commuting operators. It is obvious that $(\mathrm{C}) \subset (\mathrm{WC})$, and one may thus ask the weaker question whether

$$(\mathrm{C}) \cap (\mathrm{PB}) \subset (\mathrm{SC}). \tag{6}$$

A partial answer to this question was given in [18], and one of the main purposes of this paper is to complete the answer by establishing (6). To accomplish this, we first develop some material about tensor products of operators in the classes (PW), (PB), and (CPB).

2. Centered Operators

For use throughout the paper, we introduce the following notation and terminology. We write $\mathbb{C}$ for the complex plane, $\mathbb{D}$ for the open unit disc in $\mathbb{C}$, and $\mathbb{T}$ for $\partial\mathbb{D}$. As usual, $\mathbb{N}$ will denote the set of positive integers, $\mathbb{N}_0$ the set of nonnegative integers, and $\mathbb{Z}$ the set of integers. If $n \in \mathbb{N} \cup \{\aleph_0\}$ and $\mathscr{H}$ is any complex Hilbert space, we write $\mathscr{H}^{(n)}$ for the (orthogonal) direct sum of n copies of $\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$ we write $\sigma(T)$ and $|\sigma(T)|$ for the spectrum and spectral radius of T, respectively. If $\{D_n\}_{n=1}^{\infty}$ is any bounded sequence from $\mathscr{L}(\mathscr{H})$, we denote by $\mathrm{Diag}(D_1, D_2, ...)$ the operator in $\mathscr{L}(\mathscr{H}^{(\aleph_0)})$ satisfying

$$\mathrm{Diag}(D_1, D_2, ...)(k_1, k_2, ...) = (D_1 k_1, D_2 k_2, ...)$$

for all vectors $(k_1, k_2, ...)$ in $\mathscr{H}^{(\aleph_0)}$. Of course, $\mathrm{Diag}(D_1, D_2, ...)$ is also the direct sum $\bigoplus_{n=1}^{\infty} D_n$. Furthermore, if $\{W_n\}_{n=1}^{\infty}$ is any bounded sequence from $\mathscr{L}(\mathscr{H})$, we denote by $S_{\{W_n\}}$ the operator in $\mathscr{L}(\mathscr{H}^{(\aleph_0)})$ satisfying

$$S_{\{W_n\}}(k_1, k_2, ..., k_n, ...) = (0, W_1 k_1, W_2 k_2, ..., W_n k_n, ...) \tag{7}$$

for all vectors $(k_1, k_2, ...)$ in $\mathscr{H}^{(\aleph_0)}$. (In other words, $S_{\{W_n\}}$ is the unilateral operator-weighted shift with weight sequence $\{W_n\}$.) In the special case in which all the weights in (7) coincide with one weight W, we shall denote $S_{\{W_n\}}$ simply as $S_{\{W\}}$. Clearly $S_{\{W\}}$ is unitarily equivalent to the tensor product $S \otimes W$ acting on $\mathscr{H} \otimes \mathscr{H}$ in the usual way, where S is a unilateral shift in $\mathscr{L}(\mathscr{H})$ satisfying $Se_n = e_{n+1}$, $n \in \mathbb{N}$, for some orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $\mathscr{H}$. We begin our program with the following lemma.

LEMMA 2.1. *Suppose T and N belong to $\mathscr{L}(\mathscr{H})$, where N is a normal operator of norm one. Then $T \in (\mathbf{PB})$ (resp., $T \in (\mathrm{PW})$, $T \in (\mathbf{CPB})$) if and only if the tensor product $T \otimes N$, acting as usual on the Hilbert space $\mathscr{H} \otimes \mathscr{H}$, belongs to $(\mathrm{PB})(\mathscr{H} \otimes \mathscr{H})$ (resp., (PW), (CPB)), and one also has $M_{\mathrm{pb}}(T) = M_{\mathrm{pb}}(T \otimes N)$ (resp., $M_{\mathrm{pw}}(T) = M_{\mathrm{pw}}(T \otimes N)$, $M_{\mathrm{cpb}}(T) = M_{\mathrm{cpb}}(T \otimes N)$).*

Proof. Let $\mathscr{A}$ be the abelian unital C*-algebra generated by N, let X be the maximal ideal space of $\mathscr{A}$, and let ρ be the Gelfand transform of $\mathscr{A}$ onto $C(X)$, which is, of course, a C*-isomorphism. Let $\mathscr{L}(\mathscr{H}) \otimes_c \mathscr{A}$ and $\mathscr{L}(\mathscr{H}) \otimes_c C(X)$ be the C*-tensor product algebras, and recall (cf. [9, p. 848]) that there exists a C*-isomorphism $\tilde{\rho}$ of $\mathscr{L}(\mathscr{H}) \otimes_c \mathscr{A}$ onto $\mathscr{L}(\mathscr{H}) \otimes_c C(X)$ satisfying

$$\tilde{\rho}\left(\sum_{k=1}^{n} R_k \otimes A_k\right) = \sum_{k=1}^{n} R_k \otimes \rho(A_k)$$

for every $n \in \mathbb{N}$ and every pair of sequences $\{R_k\}_{k=1}^{n} \subset \mathscr{L}(\mathscr{H})$ and $\{A_k\}_{k=1}^{n} \subset \mathscr{A}$. Moreover, there exists another C*-isomorphism φ of $\mathscr{L}(\mathscr{H}) \otimes_c C(X)$ into the C*-algebra $C(X, \mathscr{L}(\mathscr{H}))$ of all continuous functions from X to $\mathscr{L}(\mathscr{H})$ under the supremum norm such that

$$\varphi\left(\sum_{k=1}^{n} R_k \otimes \rho(A_k)\right) = F,$$

where

$$F(x) = \sum_{k=1}^{n} (\rho(A_k)(x))\, R_k, \qquad x \in X$$

(cf. [9, p. 849]). We write $G = (\varphi \circ \tilde{\rho})(T \otimes N)$, so $G(x) = \omega(x)\, T$, $x \in X$, where $\omega = \rho(N)$. Clearly $T \otimes N \in (\mathrm{PB})$ (resp., (PW), (CPB)) if and only if G is in the same class, and in this case $M_{\mathrm{pb}}(T \otimes N) = M_{\mathrm{pb}}(G)$ and similarly for the power bounds and complete polynomial bounds. Suppose now that $T \in (\mathbf{PB})$, and let p be the polynomial $p(\zeta) = \sum_{k=0}^{n} a_k \zeta^k$. Then

$$\|p(G)\| = \sup_{x \in X} \left\| \sum_{k=0}^{n} a_k G^k(x) \right\| = \sup_{x \in X} \left\| \sum_{k=0}^{n} a_k \omega^k(x)\, T^k \right\|. \tag{8}$$

For each $x \in X$, let q_x be the polynomial

$$q_x(\zeta) = \sum_{k=0}^{n} a_k \omega^k(x)\, \zeta^k,$$

and note that from (8),

$$\|p(G)\| = \sup_{x\in X} \|q_x(T)\| \leqslant \sup_{x\in X} M_{\rm pb}(T)\, \|q_x\|_\infty = M_{\rm pb}(T) \sup_{x\in X}\ \sup_{\zeta\in\mathbb{T}} |q_x(\zeta)|$$

$$= M_{\rm pb}(T) \sup_{x\in X}\ \sup_{\zeta\in\mathbb{T}} \left|\sum_{k=0}^{n} a_k \omega^k(x)\, \zeta^k\right|.$$

Since $\|N\| = 1$ by hypothesis, $|\omega(x)| \leqslant 1$ on X and thus

$$\|p(G)\| \leqslant M_{\rm pb}(T) \sup_{\lambda\in\mathbb{D}^-} \left|\sum_{k=0}^{n} a_k \lambda^k\right| = M_{\rm pb}(T)\, \|p\|_\infty.$$

Thus G is polynomially bounded and satisfies $M_{\rm pb}(G) \leqslant M_{\rm pb}(T)$.

On the other hand, suppose now that G is polynomially bounded, let $x_0 \in X$ be such that $|\omega(x_0)| = 1$ and write $c_k = a_k \bar{\omega}^k(x_0)$, $k = 0, ..., n$. Then,

$$\|p(T)\| = \left\|\sum_{k=0}^{n} a_k T^k\right\| = \left\|\sum_{k=0}^{n} c_k \omega^k(x_0)\, T^k\right\| \leqslant \sup_{x\in X} \left\|\sum_{k=0}^{n} c_k \omega^k(x)\, T^k\right\|$$

$$= \sup_{x\in X} \left\|\sum_{k=0}^{n} c_k G^k(x)\right\| = \left\|\sum_{k=0}^{n} c_k G^k\right\|.$$

Upon setting $r(\zeta) = \sum_{k=0}^{n} c_k \zeta^k$, we obtain

$$\|p(T)\| \leqslant M_{\rm pb}(G)\, \|r\|_\infty = M_{\rm pb}(G) \sup_{\zeta\in\mathbb{T}} \left|\sum_{k=0}^{n} c_k \zeta^k\right|$$

$$= M_{\rm pb}(G) \sup_{\zeta\in\mathbb{T}} \left|\sum_{k=0}^{n} a_k (\bar{\omega}(x_0)\, \zeta)^k\right| = M_{\rm pb}(G)\, \|p\|_\infty,$$

which shows that $T \in (\mathrm{PB})$ and that the polynomial bound of T satisfies $M_{\rm pb}(T) \leqslant M_{\rm pb}(G)$. Thus all of the assertions of the lemma concerning polynomial boundedness are true, and the arguments concerning power boundedness and complete polynomial boundedness follow from similar calculations and are thus omitted. ∎

COROLLARY 2.2. *If T and U belong to $\mathscr{L}(\mathscr{H})$ with U unitary, then $T \in (\mathrm{PB})$ (resp., $T \in (\mathrm{PW})$, $T \in (\mathrm{CPB})$) if and only if $T \otimes U \in (\mathrm{PB})$ (resp., $T \otimes U \in (\mathrm{PW})$, $T \otimes U \in (\mathrm{CPB})$), and the corresponding bounds for T and $T \otimes U$ are the same.*

We can now prove our first theorem.

THEOREM 2.3. *Suppose T and C belong to $\mathscr{L}(\mathscr{H})$, where $\|C\| = |\sigma(C)| = 1$. Then $T \in (\mathrm{PB})$ (resp., $T \in (\mathrm{PW})$, $T \in (\mathrm{CPB})$) if and only if*

$T \otimes C \in (\mathrm{PB})$ (resp., $T \otimes C \in (\mathrm{PW})$, $T \otimes C \in (\mathrm{CPB})$), and, furthermore, the corresponding bounds for T and $T \otimes C$ are the same.

Proof. Suppose first that $T \in (\mathrm{PB})$. Since C is a contraction, there exists a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a unitary operator $U \in \mathscr{L}(\mathscr{K})$ such that $\mathscr{H}$ is semi-invariant for U and the compression $U_{\mathscr{H}}$ of U to $\mathscr{H}$ is C. It follows easily that $\mathscr{H} \otimes \mathscr{H} \subset \mathscr{H} \otimes \mathscr{K}$ is a semi-invariant subspace for $T \otimes U$ and that the compression $(T \otimes U)_{\mathscr{H} \otimes \mathscr{H}} = T \otimes C$. Since $T \otimes U \in (\mathrm{PB})$ by Corollary 2.2, and T and $T \otimes U$ have the same polynomial bound, it is clear that the compression $T \otimes C \in (\mathrm{PB})$ and satisfies $M_{\mathrm{pb}}(T \otimes C) \leqslant M_{\mathrm{pb}}(T)$.

To prove the other implication, suppose that $T \otimes C \in (\mathrm{PB})$. Since (PB) is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$, there is no loss of generality in supposing that $1 \in \sigma(C)$. Since $\mathbb{T} \cap \sigma(C) \subset \partial\sigma(C)$, either 1 must be an eigenvalue of C or $1 \in \sigma_{le}(C)$, the left essential spectrum of C. In the former case, T is the restriction to an invariant subspace of $T \otimes C$, and the conclusions that $T \in (\mathrm{PB})$ and $M_{\mathrm{pb}}(T) \leqslant M_{\mathrm{pb}}(T \otimes C)$ follow immediately. In the latter case, there exists an orthonormal sequence $\{y_n\}$ in $\mathscr{H}$ such that $\|Cy_n - y_n\| \to 0$ and, hence,

$$\|C^k y_n - y_n\| \to 0, \qquad k \in \mathbb{N}. \tag{9}$$

Now let $p(\zeta) = \sum_{k=1}^{m} a_k \zeta^k$ be any fixed polynomial, and let ε be an arbitrary positive number. Choose a unit vector x in $\mathscr{H}$ such that $\|p(T)x\| > \|p(T)\| - \varepsilon/2$. Now consider

$$\begin{aligned} M_{\mathrm{pb}}(T \otimes C)\,\|p\|_\infty \geqslant \|p(T \otimes C)(x \otimes y_n)\| &= \left\| \left(\sum_{k=1}^{m} a_k (T^k \otimes C^k) \right) (x \otimes y_n) \right\| \\ &= \left\| \sum_{k=1}^{m} (a_k T^k x \otimes C^k y_n) \right\| \sim \left\| \sum_{k=1}^{m} (a_k T^k x \otimes y_n) \right\| \\ &= \|p(T)\,x \otimes y_n\| = \|p(T)\,x\| \geqslant \|p(T)\| - \varepsilon/2, \end{aligned}$$

so by virtue of (9) we have

$$M_{\mathrm{pb}}(T \otimes C)\,\|p\|_\infty \geqslant \|p(T)\| - \varepsilon,$$

provided n is chosen large enough. Since ε was arbitrary, this shows that $T \in (\mathrm{PB})$ and that $M_{\mathrm{pb}}(T) \leqslant M_{\mathrm{pb}}(T \otimes C)$. Thus all of the conclusions of the theorem concerning polynomial boundedness have been established. The corresponding arguments concerning power boundedness and complete polynomial boundedness are easy modifications of the above arguments and are omitted. ∎

The following corollary is very useful.

COROLLARY 2.4. *If S and T belong to $\mathscr{L}(\mathscr{H})$ and S is a unilateral shift operator, then T belongs to one of the classes* (PB), (PW), (CPB), *if and only if $T \otimes S$ (resp., $S_{\{T\}}$ (in the notation introduced above)) belongs to the same class, and the corresponding bounds for T and $S_{\{T\}}$ are identical.*

As a first application of Corollary 2.4 we obtain the following interesting result.

PROPOSITION 2.5. *Let $T \in \mathscr{L}(\mathscr{H})$ and assume that there exists a sequence of operators $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{L}(\mathscr{H})$ satisfying $\|A_{n+1}^{-1} T A_n\| \leqslant 1$ and* $\max\{\sup_n \|A_n\|, \sup_n \|A_n^{-1}\|\} \leqslant M$. *Then there exists a single operator A in $\mathscr{L}(\mathscr{H})$ such that $\|A^{-1} T A\| \leqslant 1$ and $\|A\| = \|A^{-1}\| \leqslant M$.*

Proof. By [14, Theorem 8.1] and Corollary 2.4, it suffices to show that $S_{\{T\}}$ is similar to a contraction and that $M_{\mathrm{cpb}}(S_{\{T\}}) \leqslant M^2$. A calculation using the hypotheses shows that

$$\operatorname{Diag}(A_1^{-1}, A_2^{-1}, \ldots)\, S_{\{T\}} \operatorname{Diag}(A_1, A_2, \ldots) = S_{\{A_{n+1}^{-1} T A_n\}}.$$

Moreover, $S_{\{A_{n+1}^{-1} T A_n\}}$ is obviously a contraction and

$$\|\operatorname{Diag}(A_1^{-1}, A_2^{-1}, \ldots)\|\, \|\operatorname{Diag}(A_1, A_2, \ldots)\| \leqslant M^2,$$

so $M_{\mathrm{cpb}}(S_{\{T\}}) \leqslant M^2$. ∎

The following is one of our main theorems and establishes a better result than (6).

THEOREM 2.6. *The inclusion* (C) $\cap$ (PW) $\subset$ (SC) *is valid.*

Proof. Let T be a centered and power bounded operator in $\mathscr{L}(\mathscr{H})$. Then, according to the beautiful structure theorem of Morrel and Muhly [12], T can be written as a direct sum

$$T = T_{\mathrm{I}} \oplus T_{\mathrm{II}} \oplus T_{\mathrm{III}} \oplus T_{\mathrm{IV}},$$

where T_N is a centered operator of type N, $N = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, according to the terminology of [12], and it is obvious that each $T_N \in$ (PW) along with T. Moreover, the third author showed [18, Theorem 1.1] that $T_{\mathrm{I}} \oplus T_{\mathrm{II}} \oplus T_{\mathrm{III}} \in$ (SC), and thus it suffices to show that $T_{\mathrm{IV}} \in$ (SC). The distinguishing feature of centered operators of type IV is that the partial isometry appearing in the polar decomposition of such an operator is a unitary operator. Thus we may write $T_{\mathrm{IV}} = UP$ with U unitary, and it is a property of centered operators [12, Lemma 3.1] that the sequence

$\{U^{*n}PU^n\}_{n=1}^{\infty}$ consists of mutually commuting (positive) operators. By Corollary 2.4, to show that $T_{IV} = UP \in (SC)$, it suffices to establish that $S_{\{UP\}} \in (SC)$, and we know (Corollary 2.4) that $S_{\{UP\}} \in (PW)$ with T_{IV}. A calculation shows that

$$\mathrm{Diag}(1, U^*, U^{*2}, \ldots)\, S_{\{UP\}}\, \mathrm{Diag}(1, U, U^2, \ldots) = S_{\{U^{*n}PU^n\}_{n \in \aleph_0}},$$

and since $S_{\{UP\}}$ and $S_{\{U^{*n}PU^n\}}$ are unitarily equivalent, it suffices to show that this latter operator, which is obviously power bounded, belongs to (SC). But $S_{\{U^{*n}PU^n\}}$ is an operator weighted unilateral shift with mutually commuting normal weights, and thus belongs to (SC) by [18, Theorem 1.2] or the stronger [16, Theorem 2.6]. ∎

Remark 2.7. The above theorem shows that for centered operators, power boundedness is sufficient to imply similarity to a contraction, but this cannot be true for the larger class of weakly centered operators. For, if $(WC) \cap (PW) \subset (SC)$, then, since every operator in (PW) has a dilation that is in $(WC) \cap (PW)$ (cf. the construction in [17 or 15]), it would follow that $(PW) \subset (SC)$, which we know to be false.

Remark 2.8. We also remark that the construction in the proof of Theorem 2.6 shows that it cannot be true that every power bounded operator $S_{\{N_n\}}$, where $\{N_n\}$ is a (bounded) sequence of normal operators, belongs to (SC). (For if so, then $S_{\{UP\}}$ (and UP) would belong to (SC) for every invertible UP in (PW), which is false.) Similarly, it cannot be true that every power bounded operator $S_{\{W_n\}}$, where the (bounded) sequence $\{W_n\}$ consists of mutually commuting operators, belongs to (SC). (For if so, then every $S_{\{T\}}$ (and T) that belongs to (PW) would also belong to (SC).) Thus, [16, Theorem 2.6] may be near to best possible among the theorems in which power boundedness implies similarity to a contraction.

THEOREM 2.9. *If T is a centered, power-bounded operator in $\mathcal{L}(\mathcal{H})$ and $\sigma(T) \supset \mathbb{T}$, then either T is reflexive or T has a nontrivial hyperinvariant subspace. In particular, T has nontrivial invariant subspaces.*

Proof. By Theorem 2.6, T is similar to a contraction operator C (satisfying $\sigma(C) \supset \mathbb{T}$). By [4, Corollary 7.3], C is either reflexive or has a nontrivial hyperinvariant subspace, and one knows that these properties are invariant under similarity transforms. ∎

3. Weakly Centered Operators

In this section we study weakly centered operators. The following lemma is an easy consequence of the spectral theorem and is contained in [2].

LEMMA 3.1. *If $T \in \mathscr{L}(\mathscr{H})$ with polar decomposition $T = UP$ (where $P = (T^*T)^{1/2}$), then $T \in (\mathrm{WC})$ if and only if P commutes with UPU^*. Furthermore, the class (WC) is self-adjoint and closed under multiplication by complex numbers, taking inverses, and formation of direct sums.*

As an immediate consequence of Lemma 3.1, the earlier remark that to establish (4) it suffices to establish (5), and the proof of Theorem 2.6, we have the following interesting result.

THEOREM 3.2. *Every polynomially bounded operator in $\mathscr{L}(\mathscr{H})$ is similar to a contraction if and only if every polynomially bounded operator-weighted shift $S_{\{P_n\}}$ whose weights are positive semi-definite operators satisfying $P_n P_{n+1} = P_{n+1} P_n$, $n \in \mathbb{N}$, is similar to a contraction.*

Proof. By (5) and Corollary 2.4, it is enough to consider $S_{\{T\}}$ with T weakly centered and invertible. Let $T = UP$ be the polar decomposition of T, let D be the unitary operator $\mathrm{Diag}(I, U^*, U^{*2}, \ldots)$, and note that by Lemma 3.1, $D^{-1} S_{\{T\}} D = S_{\{P_n\}}$, where each P_n is positive semi-definite and satisfies $P_n P_{n+1} = P_{n+1} P_n$. ∎

To contrast the characterization of weakly centered operators in Lemma 3.1 with that of centered operators, we recall from [12] the following.

PROPOSITION 3.3. *If $T \in \mathscr{L}(\mathscr{H})$ and is quasi-invertible with polar decomposition $T = UP$, then T is centered if and only if the infinite sequence $\{U^n P U^{*n}\}_{n \in \mathbb{N}_0}$ consists of mutually commuting operators.*

Under certain easily stated conditions, quasi-invertible weakly centered operators are centered.

PROPOSITION 3.4. *If $T \in (\mathrm{WC})$ and is quasi-invertible (so that in the polar decomposition $T = UP$, U is unitary), and $UPU^* \in \mathscr{V}$, the unital von Neumann algebra generated by P, then T is centered (and of type IV).*

Proof. Since $\mathscr{V}$ is abelian, it suffices to show that the family $\{U^n P U^{*n}\}_{n \in \mathbb{N}_0}$ is contained in $\mathscr{V}$ (because of Proposition 3.3). We argue by induction. By hypothesis, P and UPU^* belong to $\mathscr{V}$. Suppose now that for some $n \geqslant 1$, $\{U^k P U^{*k}\}_{k=0}^{n} \subset \mathscr{V}$, and let $B \in \mathscr{V}$. Then there is a net $\{p_\lambda\}$ of polynomials such that $\{p_\lambda(P)\}$ tends to B in the weak operator topology (WOT). Thus UBU^* is the WOT-limit of the net $\{p_\lambda(UPU^*)\}$, so $UBU^* \in \mathscr{V}$. Thus, in particular, $U^{n+1} P U^{*n+1} = U(U^n P U^{*n}) U^* \in \mathscr{V}$, and by induction $\{U^k P U^{*k}\}_{k \in \mathbb{N}_0} \subset \mathscr{V}$. ∎

COROLLARY 3.5. *If $T \in (\mathrm{WC})$ and is quasi-invertible with polar decomposition $T = UP$, and P has a cyclic vector, then T is centered (and of type IV).*

Proof. Since P has a cyclic vector, the unital von Neumann algebra $\mathscr{V}$ generated by P is maximal abelian. Since P commutes with UPU^* by Lemma 3.1, $UPU^* \in \mathscr{V}$, and the result follows from Proposition 3.4. ∎

This corollary was proved by S. Parrott (cf. [2, Theorem 6]).

EXAMPLE 3.6. Unfortunately, Corollary 3.5 cannot be improved by replacing the hypothesis that P has uniform multiplicity 1 by the weaker hypothesis that P has uniform finite multiplicity, as the following example shows.

Let $\mathscr{H}$ be a four-dimensional complex Hilbert space, and let $\mathfrak{E} = \{e_i\}_{i=1}^4$ be an ordered orthonormal basis for $\mathscr{H}$. Let $P \in \mathscr{L}(\mathscr{H})$ be the positive operator such that $M_{\mathfrak{E}}(P)$, the matrix for P relative to $\mathfrak{E}$, is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

and let U_1 and U_2 be the unitary operators in $\mathscr{L}(\mathscr{H})$ such that

$$M_{\mathfrak{E}}(U_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_{\mathfrak{E}}(U_2) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define $U = U_1 U_2$ and $T = UP$. Then, as easy calculations show, P commutes with UPU^* so $T \in (\mathrm{WC})$, but P does not commute with U^2PU^{*2}, so $T \notin (\mathrm{C})$ by Proposition 3.3.

The following curious property of weakly centered operators was proved in [3].

PROPOSITION 3.7. *If $T \in (\mathrm{WC})(\mathscr{H})$ and 0 is not in the interior of the numerical range of T, then T is normal.*

We would like to prove a theorem which completely determines the structure of quasi-invertible weakly centered operators up to unitary equivalence, and in connection with this project, we make the following conjecture.

Conjecture 3.8. Suppose that $U, P \in \mathscr{L}(\mathscr{H})$, where U is a unitary operator, P is a positive semidefinite operator with trivial kernel, and P commutes with UPU^*. Then there exist a maximal abelian von Neumann algebra $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ such that P, $UPU^* \in \mathscr{A}$, and a $*$-automorphism Φ of $\mathscr{A}$ such that $\Phi(P) = UPU^*$.

We can establish this conjecture in some special cases.

PROPOSITION 3.9. *If the positive operator P generates a maximal abelian von Neumann algebra or has pure point spectrum, then the conjecture is true for any pair (U, P), where U is unitary and UPU^* commutes with P.*

Proof. If P generates a maximal abelian von Neumann algebra $\mathscr{A}$, then $UPU^* \in \mathscr{A}$, since UPU^* commutes with P, and it is elementary that one may define $\Phi(A) = UAU^*$, $A \in \mathscr{A}$, to obtain an automorphism with the desired property.

Suppose now that P has pure point spectrum (i.e., the eigenvectors of P span $\mathscr{H}$). Then, since P is Hermitian, we can write $P = \bigoplus_i \lambda_i E_i$, where the λ_i are all different and the E_i are mutually orthogonal spectral projections for P with sum $1_{\mathscr{H}}$. Since UPU^* is unitarily equivalent to P, UPU^* also has pure point spectrum, and thus may be written as $\bigoplus_j \lambda_j F_j$, where the F_j $(= UE_jU^*)$ are the corresponding spectral projections of UPU^*. Since P commutes with UPU^* (Lemma 3.1), the E_i and F_j all commute, and consequently there exists an ordered orthonormal basis $\mathfrak{E} = \{e_n\}_{n \in \mathbb{N}}$ for $\mathscr{H}$ such that both $M_{\mathfrak{E}}(P)$ and $M_{\mathfrak{E}}(UPU^*)$ are diagonal matrices. Since these matrices are unitarily equivalent, there exists a permutation matrix Π such that $\Pi M_{\mathfrak{E}}(P) \Pi^* = M_{\mathfrak{E}}(UPU^*)$. Let W be the unitary operator in $\mathscr{L}(\mathscr{H})$ such that $M_{\mathfrak{E}}(W) = \Pi$, let $\mathscr{A}$ be the maximal abelian von Neumann algebra consisting of all (normal) operators $A \in \mathscr{L}(\mathscr{H})$ such that $M_{\mathfrak{E}}(A)$ is diagonal, and let Φ be the $*$-automorphism of $\mathscr{A}$ defined by $\Phi(A) = WAW^*$, $A \in \mathscr{A}$. Then clearly $\Phi(P) = UPU^*$, and the proof is complete. ∎

For pairs (U, P) for which Conjecture 3.8 is true, we obtain the desired structure theorem.

THEOREM 3.10. *Suppose that $T = UP$ is a quasi-invertible operator in (WC) with U unitary and P positive semidefinite, and suppose that Conjecture 3.8 is true for the pair (U, P). Let $\mathscr{A}$ be the maximal abelian von Neumann algebra containing P and UPU^* given by Conjecture 3.8 and let Φ be a $*$-automorphism of $\mathscr{A}$ such that $\Phi(P) = UPU^*$. Then there exist*

(1) *a homeomorphism τ of X onto X, where X is the (compact, Hausdorff) maximal ideal space of $\mathscr{A}$,*

(2) *a finite, regular, (perfect) Borel measure μ on X which satisfies $\mu \circ \tau \equiv \mu$, and*

(3) *a Hilbert space isomorphism W from $\mathscr{H}$ onto $L^2(X, \mu)$, such that*

$$WTW^{-1} = S\Theta M_{\Gamma(P)}, \tag{10}$$

where Γ is the Gelfand map of $\mathscr{A}$ onto $C(X)$, $M_{\Gamma(P)}$ is the operator on $L^2(X, \mu)$ of multiplication by the function $\Gamma(P)$ in $L^\infty(X, \mu)$, Θ is a unitary operator on $L^2(X, \mu)$ that commutes with $M_{\Gamma(P)}$, and S is the unitary operator on $L^2(X, \mu)$ defined by

$$(Sf)(x) = f(\tau(x))(d(\mu \circ \tau)/d\mu)^{1/2}(x), \qquad f \in L^2(X, \mu). \tag{11}$$

Furthermore, every operator of the form (10), where S, Θ, and Γ are as defined above is weakly centered.

Proof. This proof is patterned after the proof of [12, Theorem 3]. Since the map $\Gamma: \mathscr{A} \to C(X)$ is a C*-algebra isomorphism of $\mathscr{A}$ onto $C(X)$, it follows that $\varphi = \Gamma\Phi\Gamma^{-1}$ is a *-automorphism of $C(X)$, and, hence, there exists a homomorphism τ of X onto X such that

$$[\varphi(f)](x) = f(\tau(x)), \qquad f \in C(X), x \in X. \tag{12}$$

One knows (cf. [5, p. 253, Proposition 3]) that there exists a unitary operator $\tilde{U}$ in $\mathscr{L}(\mathscr{H})$ such that $\Phi(A) = \tilde{U}A\tilde{U}^*$ for all A in $\mathscr{A}$, and, hence, from (12) we have

$$\Gamma(\tilde{U}A\tilde{U}^*)(x) = \Gamma(A)(\tau(x)), \qquad A \in \mathscr{A}, x \in X. \tag{13}$$

Since $\mathscr{A}$ is maximal abelian, one knows (cf. [19, Lemma II.1.2]) that there exists a finite, regular, perfect, Borel measure μ on X and a Hilbert space isomorphism W of $\mathscr{H}$ onto $L^2(X, \mu)$ such that

$$WAW^{-1} = M_{\Gamma(A)}, \qquad A \in \mathscr{A}. \tag{14}$$

(To say that μ is perfect means that every equivalence class in $L^\infty(X, \mu)$ contains an element of $C(X)$.) If we set $\hat{U} = W\tilde{U}W^{-1}$ and combine (13) and (14), we obtain

$$\hat{U}M_\psi \hat{U} = M_{\psi \circ \tau}, \qquad \psi \in C(X). \tag{15}$$

We now show that the measure $\mu \circ \tau$ on X is equivalent to the measure μ. To this end, suppose first that K is a compact subset of X such that $\mu(K) = 0$. Since μ is regular, there exists a sequence $\{U_n\}_{n=1}^\infty$ of open subsets of X such that $K \subset U_n$ for each n and $\mu(U_n) \to 0$. By an easy extension of Urysohn's lemma (cf. [1, Problem 3V]), there exists a decreasing sequence $\{g_n\}$ of nonnegative functions in $C(X)$ such that $\chi_K \leqslant g_n \leqslant \chi_{U_n}$, $n \in \mathbb{N}$. Thus, of course,

$$0 \leqslant \int_X g_n^2 \, |f|^2 \, d\mu \leqslant \int_{U_n} |f|^2 \, d\mu, \qquad n \in \mathbb{N}, f \in L^2(X, \mu).$$

It follows easily from the absolute continuity of the integral that the sequence $\{M_{g_n}\}$ of multiplication operators on $L^2(X, \mu)$ converges to zero in the strong operator topology, and from (15) we deduce immediately that the sequence of operators $\{M_{g_n \circ \tau}\}$ on $L^2(X, \mu)$ also converges to zero in the strong operator topology. Since $0 \leqslant \chi_{\tau^{-1}(K)} \leqslant (g_n \circ \tau)^2$ on X, it follows that $\mu(\tau^{-1}(K)) = 0$ and thus that τ^{-1} maps compact sets of μ-measure zero to compact sets of μ-measure zero. That τ^{-1} maps every Borel set E of μ-measure zero to another such set now follows easily from the inner regularity of μ. Thus $\mu \ll \mu \circ \tau$, and a repetition of this argument with τ replacing τ^{-1} shows that $\mu \equiv \mu \circ \tau$. Consequently, the operator S given by (11) is well defined, and a calculation shows that S is a unitary operator on $L^2(X, \mu)$ satisfying

$$SM_\varphi S^* = M_{\varphi \circ \tau}, \qquad \varphi \in L^\infty(X, \mu). \tag{16}$$

Upon setting $\Theta_1 = S^* \hat{U}$, we see from (15) and (16) that the unitary operator Θ_1 commutes with the maximal abelian algebra $W\mathscr{A}W^*$ on $L^2(X, \mu)$, and, hence, that Θ_1 must have the form $\Theta_1 = M_{\varphi_1}$, where $|\varphi_1(x)| = 1$ a.e. on X. Thus,

$$W\tilde{U}W^{-1} = \hat{U} = S\Theta_1,$$

so $WTW^{-1} = WUPW^{-1} = (WUW^{-1})(WPW^{-1}) = WUW^{-1}M_{\Gamma(P)} = (W\tilde{U}W^{-1})(W\tilde{U}^*UW^{-1})\, M_{\Gamma(P)} = SM_{\varphi_1}(W\tilde{U}^*UW^{-1})\, M_{\Gamma(P)}$. Since $\tilde{U}P\tilde{U} = \Phi(P) = UPU^*$, the unitary operator $\tilde{U}^*U$ commutes with P, and, hence, the unitary operator $W\tilde{U}^*UW^{-1}$ commutes with $WPW^{-1} = M_{\Gamma(P)}$. Thus the product $M_{\varphi_1} W\tilde{U}^*UW^{-1}$ commutes with $M_{\Gamma(P)}$, and upon defining $\Theta = M_{\varphi_1} W\tilde{U}^*UW^{-1}$, we see that (10) is true.

To prove the last statement of the theorem, it suffices by Lemma 3.1 to show that $S\Theta M_{\Gamma(P)}\Theta^*S^*$ commutes with $M_{\Gamma(P)}$. But

$$S\Theta M_{\Gamma(P)}\Theta^*S^* = SM_{\Gamma(P)}S^* = M_{\Gamma(P) \circ \tau},$$

which obviously commutes with $M_{\Gamma(P)}$, so the theorem is proved. ∎

COROLLARY 3.11. *Suppose that $T \in \mathscr{L}(\mathscr{H})$ is quasi-invertible with polar decomposition $T = UP$, and P has a pure point spectrum. Then $T \in$ (WC) if and only if there exist unitary operators U_1, U_2 in $\mathscr{L}(\mathscr{H})$ and there exists an ordered orthonormal basis $\mathfrak{E} = \{e_i\}_{i=1}^\infty$ for $\mathscr{H}$ such that*

(A) $T = U_1 U_2 P$,

(B) U_2 *commutes with* P, *and*

(C) $M_{\mathfrak{E}}(P)$ *is diagonal and* $M_{\mathfrak{E}}(U_1)$ *is a permutation matrix.*

Remark 3.12. Recall that the first operator $T \in (\mathrm{PW}) \backslash (\mathrm{SC})$ was set forth by Foguel [6] and Halmos [7], and Lebow [10] showed later $T \notin (\mathrm{PB})$. We observe that the same invariant used by Foguel–Halmos to show that $T \notin (\mathrm{SC})$ can be used to show that $T \notin (\mathrm{PB})$, in view of the following proposition.

PROPOSITION 3.13. *Suppose that* $T \in (\mathrm{PB})(\mathscr{H})$, *and let* $Z(T) = \{x \in \mathscr{H} : \{T^n x\}_{n=1}^{\infty}$ *tends weakly to zero*$\}$. *Then*

$$Z(T) \cap Z(T^*)^{\perp} = (0). \tag{17}$$

Proof. First note that if T satisfies (17) and S is invertible, then $\tilde{T} = STS^{-1}$ also satisfies (17) since $Z(\tilde{T}^*) = S(Z(T))$ and $Z(\tilde{T}^*)^{\perp} = S(Z(T^*)^{\perp})$, and recall from [11] that every operator in $(\mathrm{PB})(\mathscr{H})$ is similar to an operator of the form $\hat{T} = T_{\mathrm{ac}} \oplus U_s$ acting on $\mathscr{H} = \mathscr{H}_{\mathrm{ac}} \oplus \mathscr{H}_s$, where T_{ac} is an absolutely continuous operator in (PB) and U_s is a singular unitary operator. Since T_{ac} has a weak* continuous H^{∞} functional calculus (cf., for example, [11]) and U is unitary, it is easy to see that $Z(\hat{T}) = \mathscr{H}_{\mathrm{ac}} \oplus Z(U_s) = \mathscr{H}_{\mathrm{ac}} \oplus Z(U_s^*) = Z(\hat{T}^*)$, so certainly $Z(\hat{T}) \cap Z(\hat{T}^*)^{\perp} = (0)$, and the proof is complete. ∎

Remark 3.14. We close with a couple of remarks concerning [3]. Campbell's form (I_2) in Theorem 7 is superfluous—the matrix given there is unitarily equivalent to

$$\begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix}$$

with $xy = 1$ and $x^2 + y^2 = 2 + b^2$, so is of the form (I_3). He also makes a misstatement on page 361 that 0 lies in the convex hull of the spectrum for all operators in the classes (I_1), (I_2), (I_3), etc.

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